A differentiation formula for spherical Bessel functions

2005 J. Phys. A: Math. Gen. 38 1687

(http://iopscience.iop.org/0305-4470/38/8/005)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 1687–1690 doi:10.1088/0305-4470/38/8/005

A differentiation formula for spherical Bessel functions

J Boersma[1,3] **and M L Glasser**[2]

[1] Department of Mathematics and Computing Science, Eindhoven University of Technology, Eindhoven, The Netherlands
[2] Department of Physics and Center for Quantum Device Technology, Clarkson University, Potsdam, NY 13699-5820, USA

Received 5 October 2004, in final form 23 November 2004
Published 9 February 2005
Online at stacks.iop.org/JPhysA/38/1687

Abstract
The differentiation formula

$$\left(1 - \frac{\sqrt{z^2+a^2}}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^n [z^{n-1/2}K_{n-1/2}(z)] = \left(z + \sqrt{z^2+a^2}\right)^n z^{-1/2}K_{1/2}(z)$$

is derived, where $K_{n-1/2}(z)$ is a modified spherical Bessel function and a is an arbitrary constant.

PACS number: 02.30.Gp

Introduction

We consider the class of modified spherical Bessel functions

$$f_n(z) = z^{n-1/2}K_{n-1/2}(z), \qquad n = 0, 1, 2, \ldots, \tag{1}$$

and the differential operator $D = (1/z)\,\mathrm{d}/\mathrm{d}z$. The function $K_{n+1/2}(z), n = 0, 1, 2, \ldots,$ is expressible in terms of elementary functions and $K_{-1/2}(z) = K_{1/2}(z)$; see Watson [1, form 3.71(8), (12)]. The differentiation formula

$$D^k f_n(z) = (-1)^k f_{n-k}(z), \qquad D^n f_n(z) = (-1)^n f_0(z), \tag{2}$$

is well known [1, form 3.71(5)].

In this note we generalize the second formula (2) to the following relation:

Theorem. *For* $n = 0, 1, 2, \ldots$ *one has*

$$(1 - xD)^n f_n(z) = (x + z)^n f_0(z), \tag{3}$$

with $x = \sqrt{z^2 + a^2}$ *and* a *arbitrary.*

The second formula (2) is the limiting case $a \to \infty$ of (3). As an application relation (3) is used to evaluate the integral I_n in (15).

[3] Professor Boersma passed away during the completion of this paper.

Derivation

We begin by defining the quantity

$$A(n,k) = (2k-1)!!\binom{n}{2k}, \qquad n = 0, 1, 2, \ldots, \qquad k = 0, 1, 2, \ldots, \tag{4}$$

for which $A(n, 0) = 1$, $A(n, k) = 0$ for $k > n/2$. It is easily verified that $A(n, k)$ satisfies the recurrence relation

$$A(n+1,k) = (n+2-2k)A(n,k-1) + A(n,k), \qquad k = 0, 1, 2, \ldots, [(n+1)/2] \tag{5}$$

with $A(n, -1) = 0$. In fact, expression (4) was found by solving (5).

Lemma 1. *For $n = 0, 1, 2, \ldots$ one has*

$$(xD)^n = \sum_{k=0}^{[n/2]} A(n,k)x^{n-2k}D^{n-k}. \tag{6}$$

Proof. We proceed by induction. Relation (6) is easily seen to hold true for $n = 0, 1$. Suppose (6) is valid for some $n > 0$. Then

$$\begin{aligned}(xD)^{n+1} &= (xD)\sum_{k=0}^{[n/2]} A(n,k)x^{n-2k}D^{n-k} \\ &= \sum_{k=0}^{[n/2]+1} (n+2-2k)A(n,k-1)x^{n+1-2k}D^{n+1-k} + \sum_{k=0}^{[n/2]} A(n,k)x^{n+1-2k}D^{n+1-k},\end{aligned} \tag{7}$$

where in the sum in the second line of (7) k has been replaced by $k-1$ and the term $k = 0$, which is 0, has been added.

Now, if n is even, $[n/2] = [(n+1)/2]$ and the term $k = [n/2]+1$ in the sum in the second line of (7) vanishes due to the factor $n+2-2k$. If n is odd, $[n/2]+1 = [(n+1)/2]$ and we add the term $k = [n/2]+1$ to the sum in the third line of (7); this term vanishes because $A(n, (n+1)/2) = 0$. By use of the recurrence relation (5) in (7), thus modified, we find

$$(xD)^{n+1} = \sum_{k=0}^{[(n+1)/2]} A(n+1,k)x^{n+1-2k}D^{n+1-k}, \tag{8}$$

in both cases of n even or odd. This completes the proof of relation (6) for all $n = 0, 1, 2, \ldots$. □

Lemma 2. *For $n = 0, 1, 2, \ldots$ one has*

$$(1-xD)^n = \sum_{m=0}^{n}\binom{n}{m}(-1)^{n-m}x^{n-m}\sum_{k=0}^{[m/2]} A(m,k)D^{n-m+k}. \tag{9}$$

Proof. By the binomial theorem

$$(1-xD)^n = \sum_{l=0}^{n}\binom{n}{l}(-1)^l(xD)^l,$$

so, by lemma 1,

$$(1-xD)^n = \sum_{l=0}^{n}\frac{n!}{(n-l)!}(-1)^l\sum_{k=0}^{[l/2]}\frac{(2k-1)!!}{(2k)!(l-2k)!}x^{l-2k}D^{l-k},$$

where the binomial coefficients have been written out explicitly.

Next, we re-arrange the sum in terms of $m = l - 2k$ and k to find

$$(1 - xD)^n = \sum_{m=0}^{n} \binom{n}{m} (-1)^m x^m \sum_{k=0}^{[(n-m)/2]} A(n-m, k) D^{m+k},$$

which on replacing m by $n - m$ gives (9). □

Proof of theorem. We first note that [1, form 3.71(1)]

$$K_{n-1/2}(z) - \frac{2n-3}{z} K_{n-3/2}(z) = K_{n-5/2}(z)$$

giving

$$f_n(z) - (2n-3) f_{n-1}(z) = z^2 f_{n-2}(z). \tag{10}$$

On the one hand, from lemma 2 and the first formula (2) we have

$$(1 - xD)^n f_n(z) = \sum_{m=0}^{n} \binom{n}{m} x^{n-m} S_m(z)$$

with

$$S_m(z) = \sum_{k=0}^{[m/2]} (-1)^k A(m, k) f_{m-k}(z). \tag{11}$$

On the other hand,

$$(x + z)^n f_0(z) = \sum_{m=0}^{n} \binom{n}{m} x^{n-m} z^m f_0(z).$$

Therefore, we must show that

$$S_m(z) = z^m f_0(z), \qquad \text{for} \quad m = 0, 1, 2, \ldots. \tag{12}$$

The latter relation is easily verified for $m = 0, 1$.

For $m \geqslant 2$ we have

$$A(m, k) - A(m-2, k) = (2m - 2k - 1) A(m-2, k-1).$$

Inserting this into (11) we find

$$S_m(z) = \sum_{k=0}^{[(m-2)/2]} (-1)^k A(m-2, k) f_{m-k}(z) + \sum_{k=0}^{[m/2]} (-1)^k (2m - 2k - 1) A(m-2, k-1) f_{m-k}(z).$$

In the second sum we replace k by $k + 1$ and obtain

$$S_m(z) = \sum_{k=0}^{[(m-2)/2]} (-1)^k A(m-2, k) [f_{m-k}(z) - (2m - 2k - 3) f_{m-k-1}(z)],$$

which, by means of (10), simplifies to

$$S_m(z) = z^2 \sum_{k=0}^{[(m-2)/2]} (-1)^k A(m-2, k) f_{m-k-2}(z) = z^2 S_{m-2}(z).$$

Consequently, (12) follows by iteration, which completes the proof of the theorem. □

Discussion

By replacing z by $-\mathrm{i}z$ in (3) it follows that for the spherical Hankel functions

$$g_n(z) = z^{n-1/2} H^{(1)}_{n-1/2}(z) \tag{13}$$

one has the differentiation formula

$$(1 + yD)^n g_n(z) = (y - \mathrm{i}z)^n g_0(z), \tag{14}$$

with $y = \sqrt{a^2 - z^2}$ and a arbitrary.

As an application of (3) we present an evaluation of the integral

$$I_n = \int_0^\infty \cos(ax - n \arctan x) K_n\big(u\sqrt{1 + x^2}\big)\, \mathrm{d}x. \tag{15}$$

Starting from [1, form 13.47(2) with $\mu = -1/2$, $\nu = n$] expressed in the form

$$J_n = \int_0^\infty \cos(ax) \frac{K_n\big(u\sqrt{1 + x^2}\big)}{(1 + x^2)^{n/2}}\, \mathrm{d}x = \sqrt{\frac{\pi}{2}} u^{-n} f_n\big(\sqrt{u^2 + a^2}\big), \tag{16}$$

we have

$$\begin{aligned} I_n &= \frac{1}{2} \int_{-\infty}^\infty (1 - \mathrm{i}x)^n\, \mathrm{e}^{\mathrm{i}ax} \frac{K_n\big(u\sqrt{1 + x^2}\big)}{(1 + x^2)^{n/2}}\, \mathrm{d}x \\ &= \left(1 - \frac{\partial}{\partial a}\right)^n J_n = \sqrt{\frac{\pi}{2}} u^{-n} \left(1 - \frac{\partial}{\partial a}\right)^n f_n\big(\sqrt{u^2 + a^2}\big). \end{aligned} \tag{17}$$

If we now write $z = \sqrt{u^2 + a^2}$, then $\partial/\partial a = aD$ and by (3)

$$I_n = \sqrt{\frac{\pi}{2}} u^{-n} (a + z)^n f_0(z) = \frac{\pi}{2} \left(\frac{a + \sqrt{u^2 + a^2}}{u}\right)^n \frac{\mathrm{e}^{-\sqrt{u^2+a^2}}}{\sqrt{u^2 + a^2}}. \tag{18}$$

The integral I_n appears, in various forms, in diffraction theory and other areas; see [2, 3]. The evaluation in [3], which is believed to be the earliest one, is more complicated.

Acknowledgment

MLG thanks Dr F Goodman for bringing the integral I_n to his attention, and the NSF for support under Grant DMR-0121146.

References

[1] Watson G N 1958 *A Treatise on the Theory of Bessel Functions* 2nd edn (Cambrige: Cambrige University Press)
[2] Bertsch G F, Esbensen H and Reed B W 1998 *Phys. Rev.* B **58** 14031
[3] Reiche F 1953 On diffraction by an infinite grating *Research Report* EM-61, Division of Electromagnetic Research, Institute of Mathematical Sciences, New York University